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The effect of aspect ratio on longitudinal diffusivity in rectangular channels

By P. C. CHATWIN

Department of Applied Mathematics and Theoretical Physics,
The University of Liverpool, U.K.

AND PAUL J. SULLIVAN

Department of Applied Mathematics,
The University of Western Ontario, Canada

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In a recent paper Doshi, Daiya & Gill (1978) showed that the value of Taylor's longitudinal diffusivity D for laminar flow in a channel of rectangular cross-section of breadth a and height b is about $8D_0$ for large values of the aspect ratio a/b, where D_0 is the value of the longitudinal diffusivity obtained by ignoring all variation across the channel. This superficially surprising result is confirmed by an independent method, and is shown to be caused by the boundary layers on the side walls of the channel. The primary purpose of the paper, however, is to consider the value of D in turbulent flow in a flat-bottomed channel of large aspect ratio, for which arguments based on physics are adduced in support of the formula $D \approx D_0[1+B]\,[1-\gamma(b/a)]$, where B and γ are positive constants independent of b/a. It is shown that this result is consistent with laboratory experiments by Fischer (1966). The paper concludes with a discussion of the practical effects of aspect ratio on longitudinal dispersion in channels whose cross-section is approximately rectangular.

1. Introduction

Taylor (1953, 1954) showed that the spreading of a passive contaminant along the axis of a pipe or channel of uniform cross-section can eventually be described by a diffusion equation with a longitudinal diffusivity D. The value of D depends on whether the flow is laminar or turbulent, but all calculations have given results in accordance with the formula†

$$D \propto \frac{W^2L^2}{K}. \tag{1.1}$$

In (1.1) W is the discharge velocity, L is a length characteristic of the dimensions of the cross-section and K, which has the dimensions of a diffusivity, is a measure of the intensity of the lateral mixing of contaminant. Thus, in laminar flow, K is taken as the molecular diffusivity κ, while in turbulent flow it is proportional to hw_*, where w_* is the shear velocity and h is a length characteristic of the smallest dimensions of

† In (1.1) a (normally small) additive contribution arising from the direct effect of diffusion in the longitudinal direction has been neglected, as it will be, for simplicity, throughout this paper. The value of this contribution is unaffected by the aspect ratio of the channel.

the cross-section. (Thus, in this case, K is of the order of the transverse eddy diffusivity if this is assumed to exist.)

The purpose of this paper is to investigate the value of D when the cross-section of the pipe or channel has one dimension much greater than the other. That there is a problem worth investigating is clear when one considers the value of D for laminar flow in a pipe of elliptical cross-section of area S. In this case (Aris 1956)

$$D = \frac{W^2 S}{48\pi\kappa}\left\{\frac{5+14r^2+5r^4}{12(r+r^3)}\right\}, \tag{1.2}$$

where r is the ratio of the minor axis to the major axis. For $r = 1$, the cross-section is a circle, and (1.2) reduces to Taylor's (1953) result, namely

$$D = D_0 = \frac{W^2 S}{48\pi\kappa}. \tag{1.3}$$

However, for $r \ll 1$ the ellipse has a large aspect ratio, and (1.2) shows that

$$\frac{D}{D_0} \approx \frac{1}{12r} \gg 1. \tag{1.4}$$

Much practical interest in longitudinal dispersion arises (see e.g. chap. 5 of Fischer *et al.* 1979) because of its importance in flows like those occurring in rivers, estuaries and canals. The prime concern of the present paper is with the effect of aspect ratio on longitudinal dispersion in channels whose cross-sections are essentially rectangular, that is those whose depths are approximately constant over the bulk of the cross-section. It will be seen in § 3 that these effects arise because of boundary layers on the side walls of the channel. Other essentially geometrical effects of aspect ratio occur when the depth of the channel is not approximately constant, as in the elliptical case above. While such cases will not be considered further here, it should be noted that relevant calculations are given by Smith (1980, 1981*b*) for several shapes of channel, including those with parabolic and triangular-shaped cross-sections.

2. Laminar flow through a channel of rectangular cross-section

Consider the value of D for laminar flow through a conduit of rectangular cross-section with breadth a and height b as shown in figure 1. It will be supposed throughout that $a > b$, with primary interest in the case $a \gg b$. The origin of axes is taken in the bottom left-hand corner, with x measured along the breadth. Under an axial pressure gradient $-G$, the axial component of fluid velocity satisfies

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu}, \tag{2.1}$$

with boundary conditions

$$w = 0 \quad \text{at} \quad x = 0, a; \quad y = 0, b. \tag{2.2}$$

By symmetry the results obviously apply also to flow in an open channel of breadth a and depth $h = \frac{1}{2}b$.

Consider first the case of an infinitely wide conduit in which there is no dependence

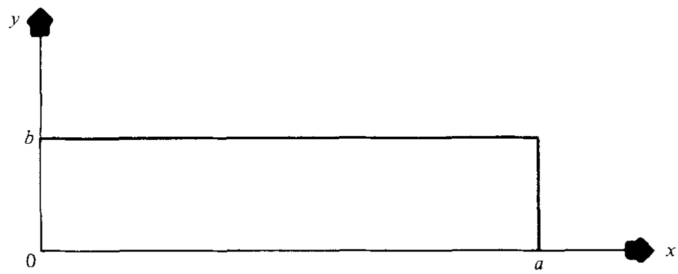

FIGURE 1. Definition sketch for the channel of rectangular cross-section.

on x, and the conditions at $x = 0, a$ in (2.2) are deleted. The axial component of velocity $w_0(y)$ and its mean with respect to y, W_0, then satisfy

$$w_0 = \frac{G}{2\mu}(by - y^2); \quad W_0 = \frac{Gb^2}{12\mu}. \tag{2.3}$$

Let D_0 be the value of the longitudinal diffusivity in this case. Its value has been calculated on many occasions (see e.g. Dewey & Sullivan 1979) with the result

$$D_0 = \frac{W_0^2 b^2}{210\kappa}. \tag{2.4}$$

This is consistent with (1.1) with L proportional to b.

Now consider the case when a/b is not infinite. The solution of (2.1) and (2.2) can be found as a Fourier series in either x or y. The most convenient form for present purposes is

$$w(x,y) = w_0(y) - \frac{4Gb^2}{\mu\pi^3} \sum_{n\,\mathrm{odd}} \frac{\cosh[n\pi(x - \frac{1}{2}a)/b]}{n^3 \cosh[n\pi a/2b]} \sin\frac{n\pi y}{b}, \tag{2.5}$$

whose cross-sectional mean W is easily found to satisfy

$$W = W_0 \left\{1 - \frac{192}{\pi^5}\left(\frac{b}{a}\right) \sum_{n\,\mathrm{odd}} \frac{\tanh[n\pi a/2b]}{n^5}\right\}. \tag{2.6}$$

The value of D in this case has been calculated by Doshi, Daiya & Gill (1978). To confirm their result, D was calculated both by means of Taylor's (1953) method, and by means of Dewey & Sullivan's (1979) method (which is based on constructing the probability density function of the cross-sectional position of a fluid molecule). Because of the zero-flux boundary condition on the distribution of concentration, the velocity profile $w(x,y)$ in (2.5) has to be re-expanded in a double Fourier series in $\cos(p\pi x/a)\cos(q\pi y/b)$. Accordingly the calculation of D, although straightforward in principle, is rather complicated and details are given in the appendix. The results of all three calculations agree and the final exact result is given by (A 10) and (A 11).

From this exact result, the value of D when $a \gg b$ can be obtained in the form

$$D = D_0[1 + A][1 - O(b/a)], \tag{2.7}$$

where D_0 is the parallel-plate value of D given in (2.4), and A is a numerical constant – independent of b/a – given by

$$A = \frac{2^{11} \times 315}{\pi^{10}} \Big(\sum_{n\,\mathrm{odd}} n^{-5}\Big)^2 \approx 6{\cdot}9512, \tag{2.8}$$

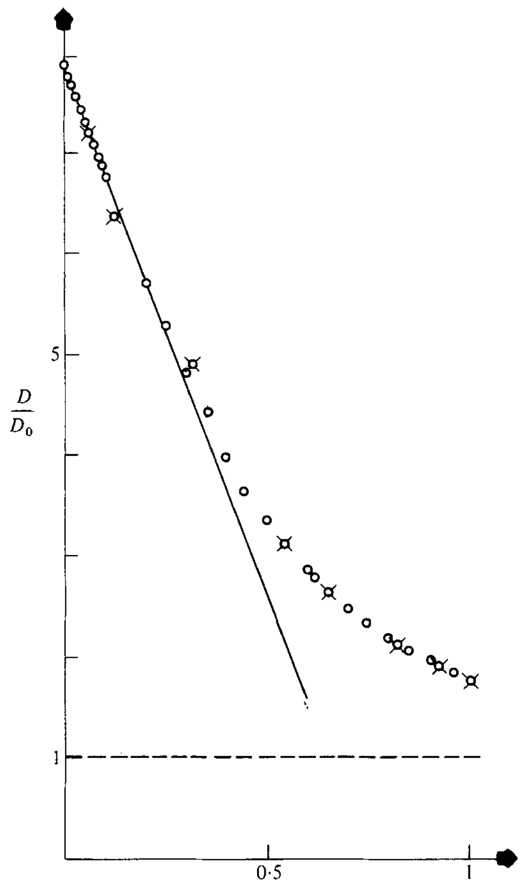

FIGURE 2. The variation of D/D_0 with b/a for laminar flow through the channel of rectangular cross-section. The open circles (○) show values calculated directly from (A 10) and (A 11), the crosses (×) show values calculated by the simulation technique of Sullivan (1971) and the straight line is that obtained from (2.7) and (2.8).

where table 23.3 of Abramowitz & Stegun (1964) has been used to obtain the numerical value of A. Figure 2 shows the variation of D with b/a, and confirms that (2.7) is correct for $b \lesssim 0{\cdot}2a$.

3. Physical explanation for the value of D

It is at first sight surprising, perhaps, that, as $a/b \to \infty$, D does not approach the value D_0 given in (2.4), obtained by ignoring all variations across the conduit, but approaches a value of about $8D_0$. After obtaining the value of D, Doshi *et al.* (1978) showed that the factor A in (2.7) could be obtained using a calculation procedure proposed by Fischer (1966). In this procedure, designed originally for use with turbulent flows in broad natural streams, it is assumed that variations of the velocity over the depth can be ignored, in the sense that $w(x,y)$ can be replaced by $w(x)$, its mean with respect to y. Then it is proposed that D can be determined by Taylor's (1953) method, given in the appendix, with all variation with y of the distribution of concentration ignored. As noted above, this procedure gives, for the present geometry, $D \to AD_0$ as $a/b \to \infty$ and is therefore about $12\frac{1}{2}$ % less than the correct value of $(1+A)D_0$; this is nevertheless a marked improvement on the superficially obvious assumption that $D \to D_0$ as $a/b \to \infty$!

While the success of Fischer's procedure in predicting the value of A clearly shows the importance of lateral velocity variations, it is not obvious physically how these will affect the value of D/D_0, or even its order of magnitude, for more complicated geometries with large aspect ratios. Thus, while the length L in (1.1) is proportional to b in the present case, Fischer's proposal would seem to allow the possibility that, in other circumstances, it may be proportional to e.g. a or $(ab)^{\frac{1}{2}}$. In view of the widespread occurrence of flows with large aspect ratios, it seems to be important to understand more about the physical reasons for the value of D/D_0 in (2.7).

When $a \gg b$, $w(x,y)-w_0(y)$ is, according to (2.5), exponentially small everywhere in the cross-section except in layers of thickness δ adjoining the side walls $x=0$ and $x=a$. It is obvious physically, and also mathematically from (2.5), that δ is of order b. Without loss of generality δ can be chosen as equal to b, since $w(x,y) \approx 0{\cdot}96w_0(y)$ when $x=b$ and $x=a-b$. Also, from (2.6), it follows that

$$W \approx W_0[1-0{\cdot}63b/a] \quad (a \gg b). \tag{3.1}$$

Now define $\omega(x,y)$ by

$$\omega(x,y) = w(x,y) - W. \tag{3.2}$$

It follows that, to an adequate approximation,

$$\omega(x,y) \approx [w_0(y)-W_0] + 0{\cdot}63(b/a)W_0 \quad (b < x < a-b). \tag{3.3}$$

By definition, $\omega(x,y)$ has zero mean over the whole cross-section. Therefore

$$\omega(x,y) \approx -[(a-2b)/a]W_0 v(x,y) \quad (0 \leqslant x < b, \quad a-b < x \leqslant a), \tag{3.4}$$

where

$$\int_0^b \int_0^b v(x,y)\,dx\,dy \approx 0{\cdot}63b^2, \tag{3.5}$$

so that $v(x,y)$ is of order 1.

Consider a contaminant molecule released from $x=X$, $y=Y$, $z=0$ at $t=0$. With respect to axes moving with the discharge velocity W, let its axial component of velocity at time t after release be $\Omega(t;X,Y)$. By extending Taylor's (1921) classic analysis to the case of an initial distribution of contaminant molecules that is uniform over $z=0$, it has been shown (Saffman 1960; Chatwin 1977; Dewey & Sullivan 1979) that

$$D = \frac{1}{ab}\int_0^\infty dt \iint \overline{\Omega(0;X,Y)\,\Omega(t;X,Y)}\,dX\,dY, \tag{3.6}$$

where the overbar denotes an ensemble mean. Dewey & Sullivan (1979) showed further that, in practice, (3.6) is the same as

$$D = \frac{1}{ab}\int_0^\infty dt \iint \omega(X,Y)\,\overline{\Omega(t;X,Y)}\,dX\,dY, \tag{3.7}$$

where $\omega(x,y)$ is given by (3.2), (3.3) and (3.4), and, as explained in an earlier footnote, a small additive contribution to D has been ignored in passing from (3.6) to (3.7).

Denote by $p(x,y,t;X,Y)$ the probability density function of the position in the cross-section at time t of a contaminant molecule released from $x = X, y = Y$ at $t = 0$. Then $\overline{\Omega(t; X, Y)}$ in (3.7) satisfies (Chatwin 1977; Dewey & Sullivan 1979)

$$\overline{\Omega(t; X, Y)} = \iint \omega(x, y)\, p(x, y, t; X, Y)\, dx\, dy. \tag{3.8}$$

Dewey & Sullivan showed how $p(x, y, t; X, Y)$ could be determined exactly for the present geometry, but that result is not needed here.

First of all, it is obvious that each contaminant molecule eventually forgets where it started from. Thus, for all X and Y,

$$\lim_{t\to\infty} p(x, y, t; X, Y) = 1/ab. \tag{3.9}$$

But a contaminant molecule forgets about its value of Y in a time of order b^2/κ, much more quickly than it forgets about its value of X. Hence

$$p(x, y, t; X, Y) \approx \frac{1}{b} p(x, t; X) \quad (t \gtrsim b^2/\kappa), \tag{3.10}$$

where

$$p(x, t; X) \approx 1/a \quad (t \gtrsim a^2/\kappa). \tag{3.11}$$

Now consider the dependence of $\overline{\Omega(t; X, Y)}$, defined in (3.8), on t for a molecule released in the central region, so that $b < X < a-b$. It follows from (3.3), (3.10) and (3.11) that

$$\overline{\Omega(t; X, Y)} \approx \left\{ \begin{array}{ll} \omega(X, Y) & (0 \leqslant t \ll b^2/\kappa), \\ 0{\cdot}63(b/a)\, W_0 & (b^2/\kappa \lesssim t \lesssim (a-2b)^2/\kappa), \\ 0 & (t \gtrsim a^2/\kappa). \end{array} \right\} \tag{3.12}$$

The basic physical reason behind these results is that it takes, on the average, a time of order $(a-2b)^2/\kappa$ for a typical contaminant molecule released in the central region to reach, and begin to sample, the layers of thickness b adjoining the side walls. In the meantime, its average axial velocity is of order $0{\cdot}63(b/a)\, W_0$, the non-zero discharge velocity in the central region. From (3.7), it now follows that the contribution to D from molecules released in the central region is of order

$$\frac{(a-2b)\, b}{ab} \left[\alpha_1 \frac{W_0^2 b^2}{\kappa} + \alpha_2 \left(\frac{b}{a} W_0 \right)^2 \frac{(a-2b)^2}{\kappa} \right], \tag{3.13}$$

where α_1 and α_2 are constants of order unity. The first term comes from the first time interval in (3.12), and hence is of order D_0. This contribution can therefore be written

$$D_0(1 + A_1)\, [1 - O(b/a)], \tag{3.14}$$

where A_1 is a constant of order unity whose value is determined predominantly by the second term in (3.13), therefore by the small non-zero discharge velocity which a typical contaminant molecule has for $t \lesssim (a-2b)^2/\kappa$.

An entirely similar analysis to the above can be carried out for molecules released in the layers of thickness b adjoining the side walls. The intermediate results corresponding to (3.12) need not be given, but the contribution to D for such molecules is found to be of order

$$\frac{b^2}{ab} \left[\beta_1 \left(\frac{a-2b}{a} W_0 \right)^2 \frac{b^2}{\kappa} + \beta_2 \left(\frac{b}{a} W_0^2 \right) \frac{(a-b)^2}{\kappa} \right], \tag{3.15}$$

where β_1 and β_2 are constants of order unity. The second term is of order D_0 and comes from times t satisfying $b^2/\kappa \lesssim t \lesssim (a-b)^2/\kappa$, during which the molecule is likely to have wandered away from the side layers into the central region where its axial velocity is non-zero. Then (3.15) reduces to

$$D_0 A_2[1-O(b/a)], \tag{3.16}$$

where A_2 is a constant of order unity. The sum of the contributions in (3.14) and (3.16) gives a result consistent with (2.7) where $A = A_1 + A_2$.

The preceding analysis makes clear that the reason why D is significantly greater than D_0, but of the same order, is that the presence of the side layers causes the mean axial velocity in the central region to be greater than the discharge velocity, by an amount proportional to b/a.

4. Turbulent flow in rectangular channels

Practically, the value of D in turbulent flows is, of course, much more important than its value in laminar flows. However, the physical arguments in § 3 are also applicable in turbulent flows. Provided the Reynolds number $R_* = w_* b/\nu$, where w_* is the shear velocity, is sufficiently large, the effects of the side walls on turbulent flow in a wide rectangular channel will be confined to layers of thickness of order b, just as in laminar flow. The argument in § 3 can therefore be applied with κ replaced by $w_* b$. Thus

$$D \approx D_0[1+B][1-\gamma(b/a)], \tag{4.1}$$

where B and γ are constants, and D_0 is the value of D when variations of mean properties of the turbulence across the channel are ignored.

As discussed in for example Elder (1959), Chatwin (1971), Sullivan (1971) and Fischer (1973), the precise value of D_0 depends on many factors such as the value of R_*, the nature of the walls and the value of the Schmidt (or Prandtl) number, and definitive knowledge of its dependence on such factors is not yet available. However, this precise value of D_0 is not of primary importance in the present paper.

For a given cross-sectional shape, the values of the constants B and γ in (4.1) will be dependent on the value of R_* and the roughness characteristics of the walls. However, the principle of Reynolds-number similarity (Monin & Yaglom 1971, p. 299) suggests that, in a smooth-walled conduit, B and γ will be universal constants for high enough R_*. The dependence of these constants on roughness is more difficult to assess. While secondary flows are not considered explicitly in the arguments in this paper, they are expected to be confined to side-wall layers and, therefore, will not affect the structure of (4.1).

For reasons that will be discussed in § 5, very few measurements of D exist that enable the validity of (4.1) to be examined. However some confirmation that (4.1) is worth more rigorous testing can be obtained using data from Fischer (1966), taken from an open channel whose cross-section was a trapezium. The width of the trapezium could be varied and its sides were artificially roughened with stones. A summary of the results from 6 separate series (2800, 2900, 3000, 3100, 3200, 3400) each consisting of at least 7 runs is given in table 1. It should be noted that:

(*a*) the values of D quoted are those obtained by Fischer using his 'diffusive-transport' method;

Series	h (cm)	a (cm)	W (cm/s)	$\frac{h}{a}$	$\frac{Dw_*}{W^2h}$
2800	3·5	38·1	25·1	0·092	0·616
2900	4·7	38·1	45·4	0·123	0·522
3000	3·5	38·1	45·1	0·092	1·157
3100	3·5	31·7	44·4	0·110	0·603
3200	2·1	31·7	45·3	0·066	1·850
3400	2·1	19·1	46·1	0·110	0·864

TABLE 1. Data from Fischer (1966) for roughened open channels with trapezoidal cross-sections

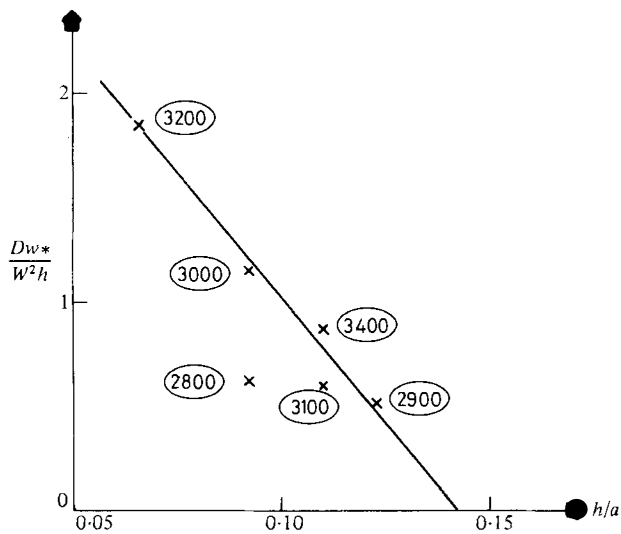

FIGURE 3. The variation of Dw_*/W^2h with h/a for experiments by Fischer (1966) in an open channel of trapezoidal cross-section.

(*b*) the values of a are the bottom widths of the trapezoidal cross-section;

(*c*) because of the atypically high measured values of w_* caused by the roughening, the values of Dw_*/W^2h in table 1 were obtained using a w_* satisfying $w_* = 0{\cdot}044\,W$, this being the average result from other series (1200, 1300, 1400, 2700) of tests by Fischer in channels without artificial roughening.

As shown in figure 3, the trend of the results in table 1 is consistent with (4.1), at least insofar as dependence on h/a is concerned. It seems therefore that the arguments in § 3, while not applying in detail to Fischer's experiments because of the trapezoidal geometry and artificial roughening, do give the correct orders of magnitude for assessing the dependence of D on aspect ratio. However the value of D_0 for Fischer's results, that is the longitudinal diffusivity that would be obtained in a calculation ignoring mean lateral variations, cannot be assessed because it would have to incorporate the artificially increased intensity of the transverse mixing and is therefore neither Elder's (1959) value nor as measured in other tests by Fischer without artificial roughening. However, it has been implicitly assumed in saying that figure 3 is consistent with (4.1) that the unknown value of D_0 is proportional to W^2h/w_*.

The straight line in figure 3, drawn by eye as a reasonable fit, has the approximate equation

$$\frac{Dw_*}{W^2h} = \frac{10}{3}\left\{1 - 7\left(\frac{h}{a}\right)\right\}. \tag{4.2}$$

For the reasons given above, the numbers $\frac{10}{3}$ and 7 depend on the shape of the cross-section and the roughness characteristics. It should also be recorded that the point in figure 3 that is furthest from the straight line (4.2) is that for the series (2800) with the lowest value of R_* ($\simeq$ 390).

5. Some remarks on the evolution of the dispersion with time

It is now well known (e.g. Chatwin 1971, 1973; Sullivan 1971; Dewey & Sullivan 1977; Fischer *et al.* 1979) that Taylor's original (1953) analysis is inadequate for dealing with most practical problems involving longitudinal dispersion. This is because, in practice, the times since release for which information is required are much less than those needed for the Taylor prescription, involving a time-independent longitudinal diffusivity D, to apply. In other words (Chatwin 1972), Taylor's result is equivalent to the assumption that the distribution of concentration is a Gaussian function of distance z along the axis of the pipe or channel, and most observed distributions exhibit marked deviations from Gaussianity. This is true, for example, for the profiles of concentration in the experiments of Fischer (1966) from which the data in table 1 were obtained. Caution is therefore needed before using the results of these, and most similar, experiments to predict absolute values of D.†

The time needed for Taylor's analysis to apply in a channel with large aspect ratio is of order a^2/κ in laminar flow, and of order a^2/bw_* in turbulent flow. In either case, the time satisfies

$$\frac{Kt}{b^2} = O\left(\frac{a^2}{b^2}\right), \tag{5.1}$$

where K is defined following (1.1). For the geometry considered in § 2, let $Z(t; X, Y)$ denote the axial co-ordinate at time t with respect to axes moving with the discharge velocity W of a contaminant molecule released from $x = X$, $y = Y$, $z = 0$ at $t = 0$. Thus $\dot{Z} = \Omega$, where $\Omega(t; X, Y)$ is the velocity introduced in § 3, and, for all X and Y,

$$\frac{1}{2}\frac{d}{dt}\overline{Z^2} \to D \tag{5.2}$$

as $t \to \infty$ (Taylor 1921). The value of $\frac{1}{2}d\overline{Z^2}/dt$ has been given by Doshi *et al.* (1978), and has also been calculated by the authors using the method of Dewey & Sullivan (1979). Figure 4 shows the values of Kt/b^2 for which $\frac{1}{2}d\overline{Z^2}/dt = 0{\cdot}90D$ and $0{\cdot}95D$. These are in agreement with (5.1).

Complementary discussions to the above have been given by Fischer (see chaps 4 and 5 of Fischer *et al.* 1979) and by Doshi *et al.* (1978). The argument may have to be amended when the flow is oscillatory (see e.g. Allen 1982). In any case, when (5.1) is not satisfied, methods different from Taylor's must be used, such as those proposed by Chatwin (1970, 1980) and Smith (1981*a*).

† For the same reason, there is little data available at present with which to test (4.1).

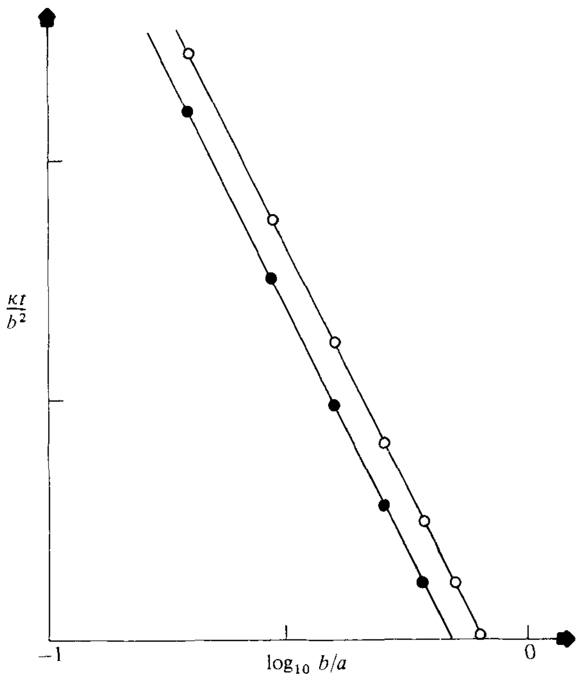

FIGURE 4. The values of $\kappa t/b^2$ for which $\frac{1}{2}d\overline{Z^2}/dt = 0{\cdot}90D$ (●) and $0{\cdot}95D$ (○). The straight lines are of slope -2, and show that the calculated points are consistent with (5.1).

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Appendix. The value of D for laminar flow in a rectangular conduit

For the flow considered in § 2, the distribution of concentration $C(x, y, z, t)$ satisfies

$$\kappa\nabla^2 C = \frac{\partial C}{\partial t} + w\frac{\partial C}{\partial z}, \tag{A 1}$$

where $w(x, y)$ is given in (2.5). The boundary conditions on C include

$$\frac{\partial C}{\partial x} = 0 \quad (x = 0, a), \quad \frac{\partial C}{\partial y} = 0 \quad (y = 0, b). \tag{A 2}$$

Taylor (1953) noted that (A 1) has an exact solution of the form

$$C = \alpha(z - Wt) + \alpha f(x, y), \tag{A 3}$$

where α is a constant, and W is the discharge velocity given in (2.6). Substitution of (A 3) in (A 1) and (A 2) gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{w - W}{\kappa}, \tag{A 4}$$

$$\frac{\partial f}{\partial x} = 0 \quad (x = 0, a), \quad \frac{\partial f}{\partial y} = 0 \quad (y = 0, b). \tag{A 5}$$

Taylor showed that D is related to f by the equation

$$D = -\frac{1}{ab}\int_0^a \int_0^b (w - W) f \, dx \, dy. \qquad \text{(A 6)}$$

In view of (A 5) it is natural to expand f and $w - W$ in Fourier series in

$$\cos(p\pi x/a)\cos(q\pi y/b),$$

where the symmetry of the flow requires that p and q are even. Thus, write

$$w - W = \sum_{\substack{p\,\mathrm{even}\\ \geqslant 2}} W_{p0}\cos\frac{p\pi x}{a} + \sum_{\substack{q\,\mathrm{even}\\ \geqslant 2}} W_{0q}\cos\frac{q\pi y}{b} + \sum_{\substack{p\,\mathrm{even}\\ \geqslant 2}}\sum_{\substack{q\,\mathrm{even}\\ \geqslant 2}} W_{pq}\cos\frac{p\pi x}{a}\cos\frac{q\pi y}{b}, \qquad \text{(A 7)}$$

$$f = \sum_{\substack{p\,\mathrm{even}\\ \geqslant 2}} F_{p0}\cos\frac{p\pi x}{a} + \sum_{\substack{q\,\mathrm{even}\\ \geqslant 2}} F_{0q}\cos\frac{q\pi y}{b} + \sum_{\substack{p\,\mathrm{even}\\ \geqslant 2}}\sum_{\substack{q\,\mathrm{even}\\ \geqslant 2}} F_{pq}\cos\frac{p\pi x}{a}\cos\frac{q\pi y}{b}. \qquad \text{(A 8)}$$

Substitution of (A 7) and (A 8) in (A 4) gives

$$-\left(\frac{p\pi}{a}\right)^2 F_{p0} = \frac{W_{p0}}{\kappa}, \quad -\left(\frac{q\pi}{b}\right)^2 F_{0q} = \frac{W_{0q}}{\kappa}, \quad -\left\{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2\right\} F_{pq} = \frac{W_{pq}}{\kappa}. \qquad \text{(A 9)}$$

Hence, from (A 6),

$$D = \frac{1}{4\kappa}\left[2\sum_{\substack{p\,\mathrm{even}\\ \geqslant 2}}\left(\frac{a}{p\pi}\right)^2 W_{p0}^2 + 2\sum_{\substack{q\,\mathrm{even}\\ \geqslant 2}}\left(\frac{b}{q\pi}\right)^2 W_{0q}^2 + \sum_{\substack{p\,\mathrm{even}\\ \geqslant 2}}\sum_{\substack{q\,\mathrm{even}\\ \geqslant 2}}\left\{\frac{W_{pq}^2}{(p\pi/a)^2 + (q\pi/b)^2}\right\}\right]. \qquad \text{(A 10)}$$

The coefficients W_{pq} are easily obtained from (2.5) using the normal integral formulae for Fourier coefficients. The results are

$$\left.\begin{aligned} W_{p0} &= -\frac{32Gab^3}{\mu\pi^5}\sum_{n\,\mathrm{odd}}\frac{\tanh(n\pi a/2b)}{n^3(n^2a^2 + p^2b^2)}, \\ W_{0q} &= -\frac{2Gb^2}{\mu\pi^2q^2}\left\{1 + \frac{8}{\pi^2}\sum_{n\,\mathrm{odd}}\frac{q^2}{n^2(n^2 - q^2)}\left[\frac{\tanh(n\pi a/2b)}{n\pi a/2b}\right]\right\}, \\ W_{pq} &= -\frac{64Gab^3}{\mu\pi^5}\sum_{n\,\mathrm{odd}}\frac{\tanh(n\pi a/2b)}{n(n^2 - q^2)}\,\frac{1}{n^2a^2 + p^2b^2}. \end{aligned}\right\} \qquad \text{(A 11)}$$

Substitution of (A 11) in (A 10) gives the value of D, and it has been checked that this agrees with the result in Doshi *et al.* (1978). The main concern of this paper is with the value of D when $b \ll a$. Under these circumstances, (A 11) can be approximated by

$$\left.\begin{aligned} W_{p0} &\approx -\frac{32Gb^3}{\mu\pi^5 a}\Big(\sum_{n\,\mathrm{odd}} n^{-5}\Big)\left[1 + O\left(\frac{b^2}{a^2}\right)\right], \\ W_{0q} &\approx -\frac{2Gb^2}{\mu q^2\pi^2}\left[1 - O\left(\frac{b}{a}\right)\right], \\ W_{pq} &\approx -\frac{64Gb^3}{\mu\pi^5p^2a}\left(\sum_{n\,\mathrm{odd}}\frac{1}{n^3(n^2 - q^2)}\right). \end{aligned}\right\} \qquad \text{(A 12)}$$

Substitution into (A 10) shows that the terms involving W_{pq} ($pq > 0$) give a contribution to D of order $D_0(b/a)^2$, which may therefore be neglected to highest order. The summation involving W_{0q} gives D_0, while that involving W_{p0} gives AD_0, where the value of A is given in (2.8). Thus (2.7) is obtained. It should be noted that the term AD_0 comes from the Fourier series across the wide dimension of the channel, consistent with the arguments in § 3.

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